IRT Potpourri

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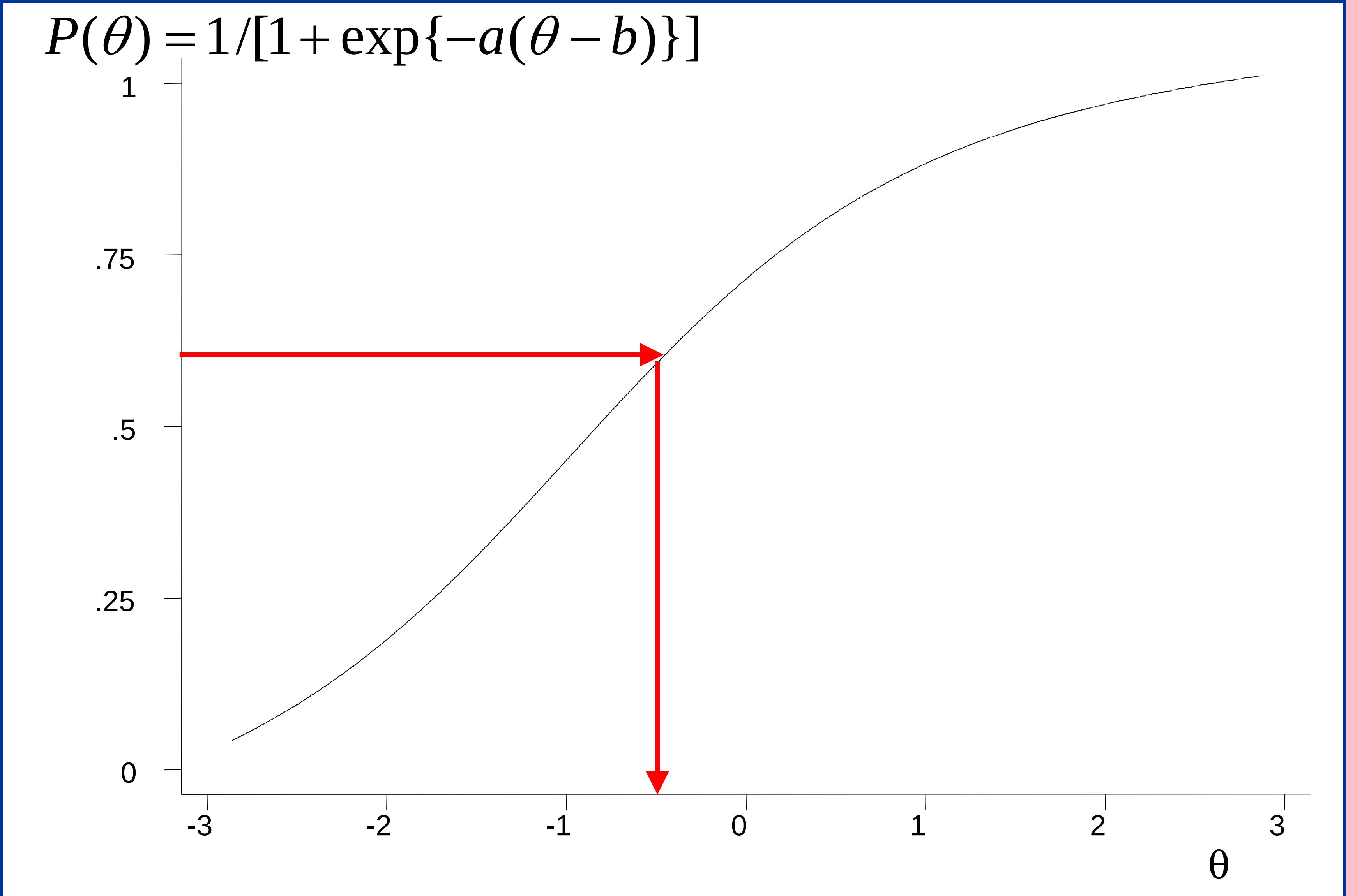
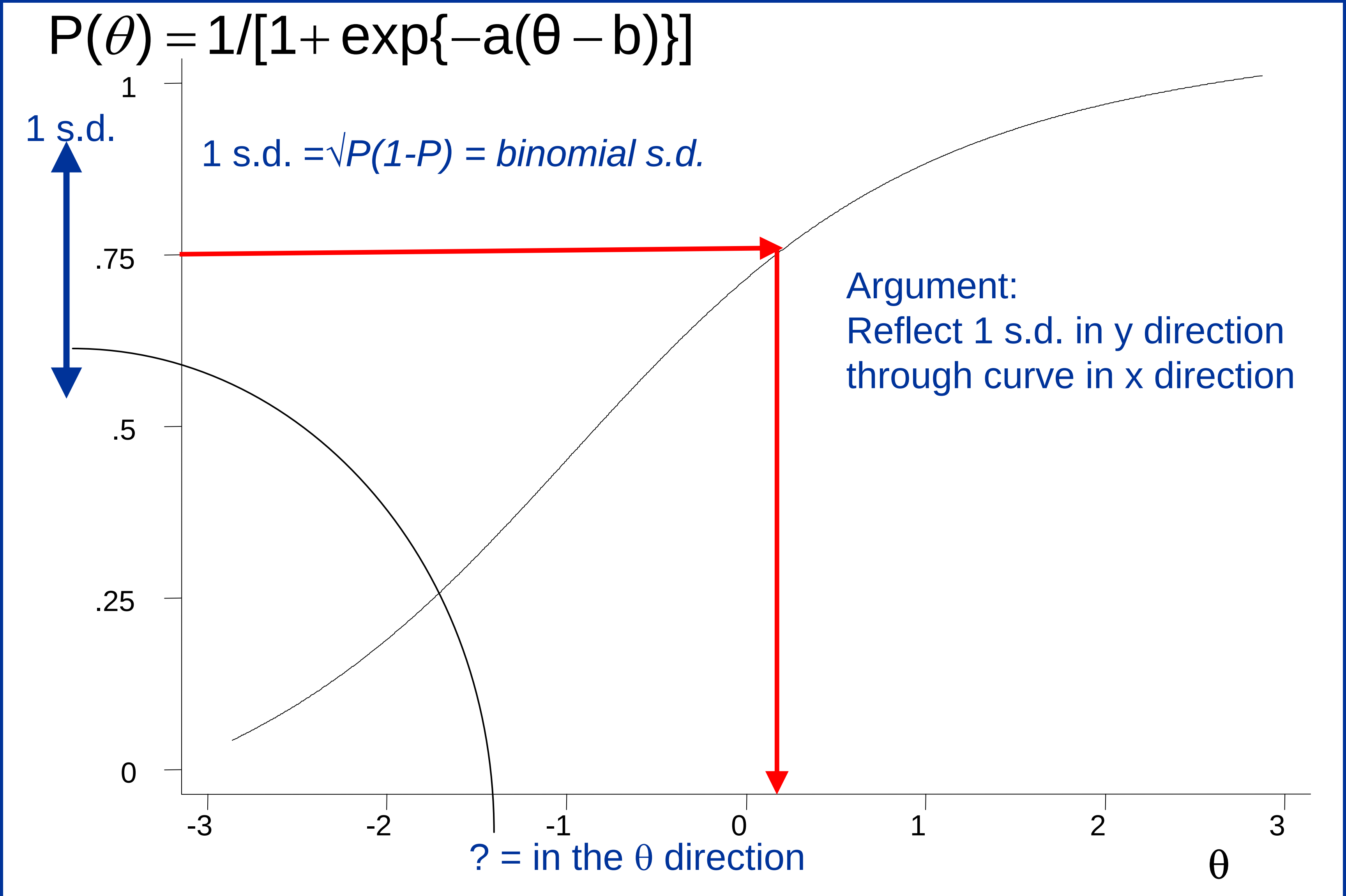
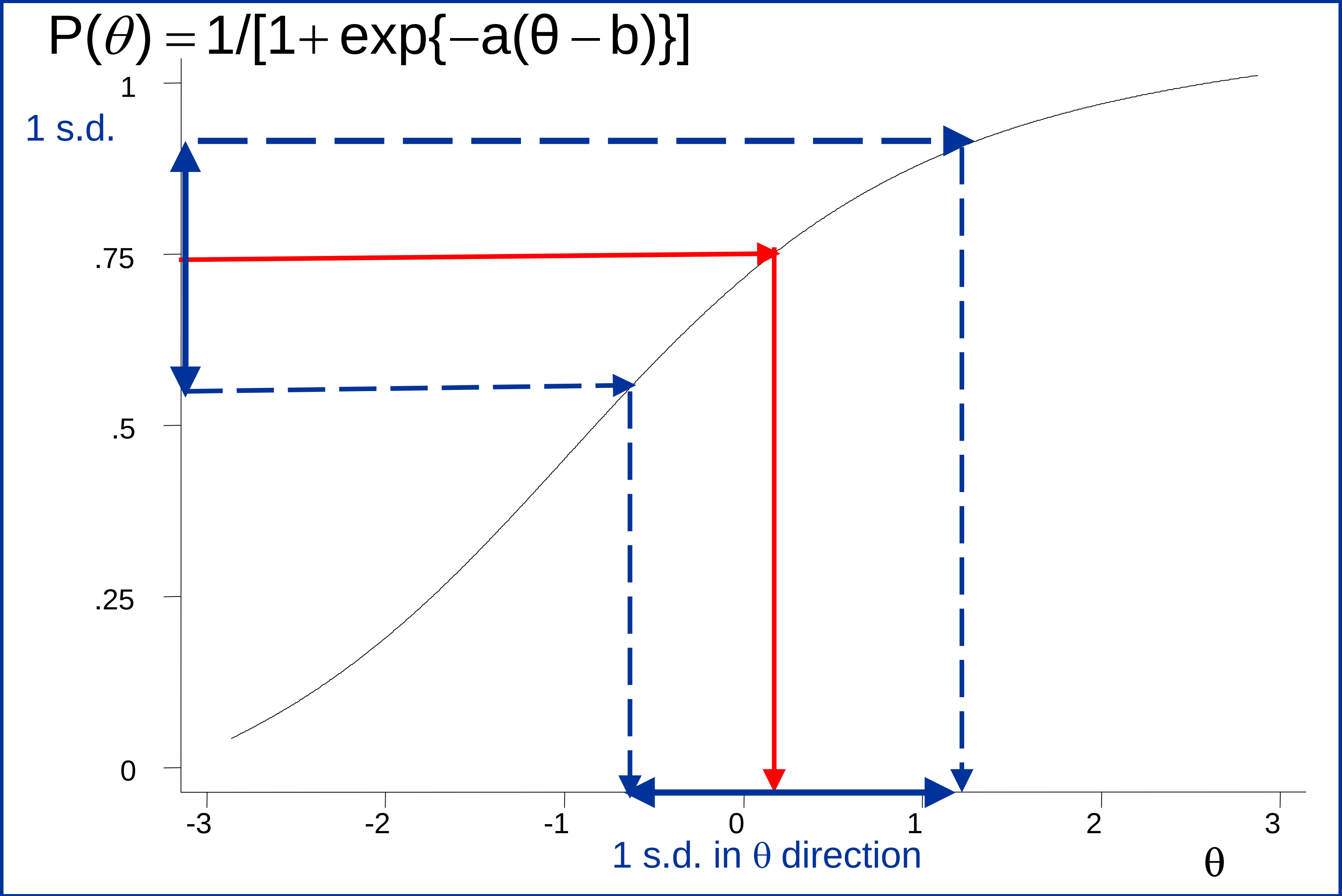
Outline

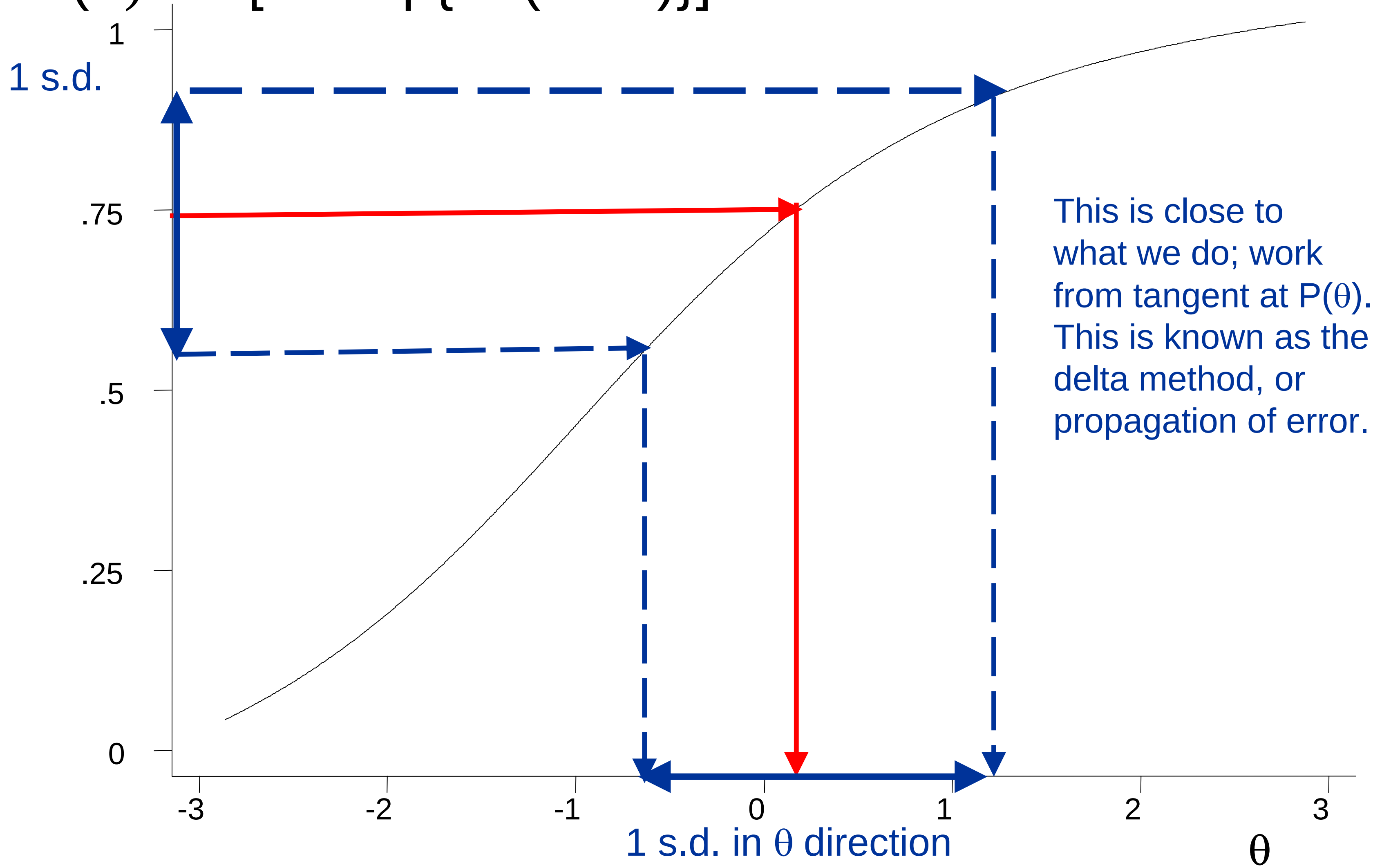
1. Geometry of information
2. Some simple results
3. IRT and link to sensitivity and specificity
4. Linear model vs IRT model—cautions
5. Measuring change—a simple model
6. Change in ability in ACT cohort


$P(\theta) = 1/[1 + \exp\{-a(\theta - b)\}]$
1
.75
.5
.25
0
-3
-2
-1
0
1
2
3
θ


$P(\theta) = 1/[1 + \exp\{-a(\theta - b)\}]$
1 s.d.
1 s.d. =√P(1-P) = binomial s.d.
1
.75
.5
.25
0
Argument:
Reflect 1 s.d. in y direction
through curve in x direction
-3
-2
-1
0
1
2
3
? = in the θ direction
θ

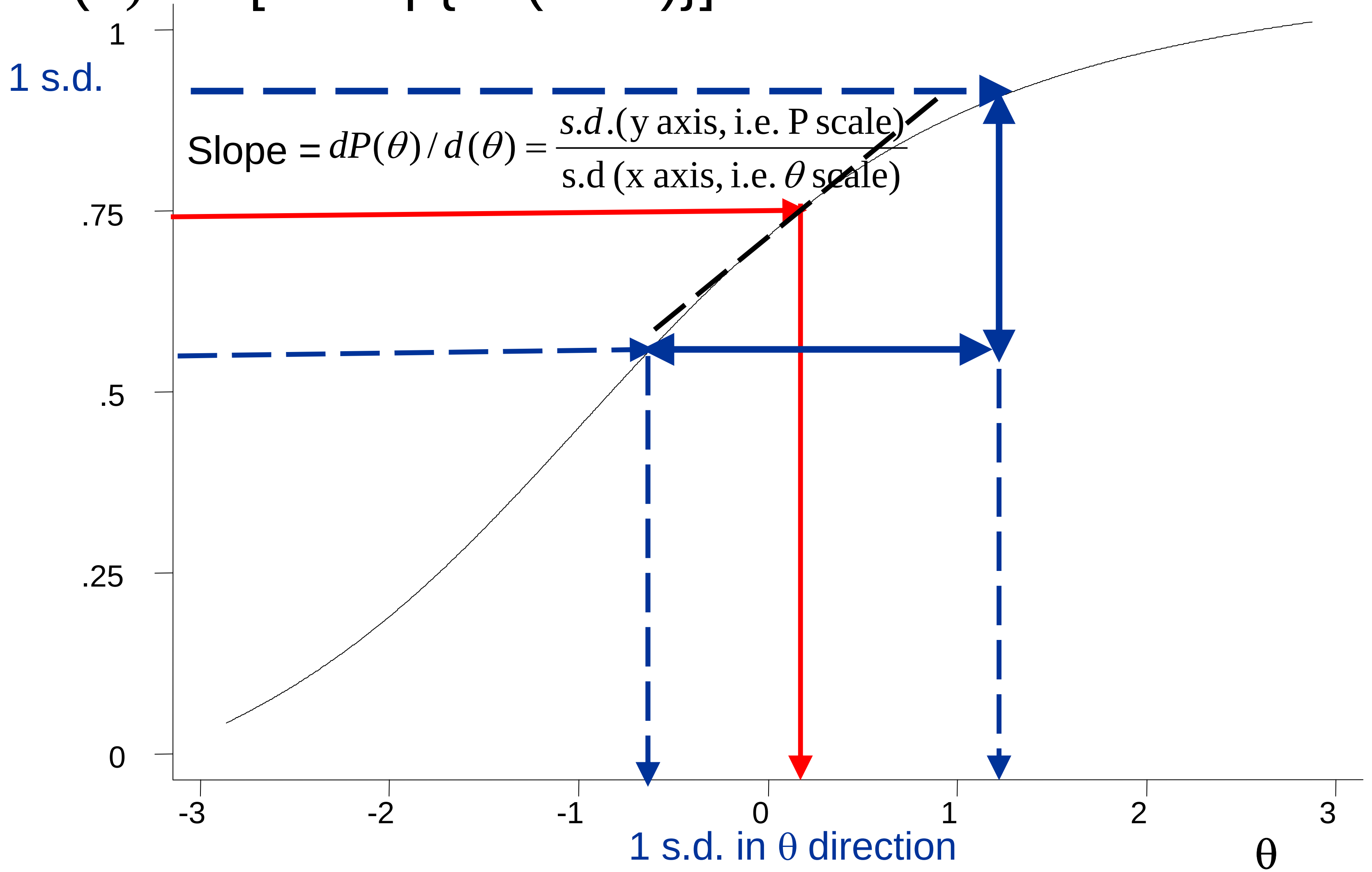

P(θ) = 1/[1+exp{−a(θ − b)}]
1
1 s.d.
.75
.5
.25
0
-3
-2
-1
0
1
2
3
1 s.d. in θ direction
θ

$$P(\theta) = 1/[1 + \exp\{-a(\theta - b)\}]$$



This is close to what we do; work from tangent at P(θ). This is known as the delta method, or propagation of error.

$$P(\theta) = 1/[1 + \exp\{-a(\theta - b)\}]$$



Coup de Grace

$$\text{Slope} = dP(\theta)/d(\theta) = \frac{\text{s.d.(y axis)}}{\text{s.d. }(\theta\text{ scale})}$$

or

$$\text{s.d. }(\theta\text{ scale}) = \frac{\text{s.d. }(\text{y axis})}{dP(\theta)/d(\theta)} = \frac{\sqrt{P(1-P)}}{dP(\theta)/d(\theta)}.$$

For the 2PL model this becomes,

$$\text{s.d.}(\theta\text{ scale}) = \frac{1}{\text{a}\sqrt{P(1-P)}}.$$

Where "a" is the discrimination of the item.

Some Simple Results

<u>Result 1 :</u>

$$I(\theta) = \frac{1}{\text{s.d.}(\theta)^2}.$$

For the logistic model this becomes,

$$I(\theta) = a^2 P(1 - P).$$

(1) Does not depend on "b," the difficulty of the item.

(2) Since $\max P(1 - P) = 1/4$,

$$I(\theta) \le \frac{a^2}{4},$$

and

$$\text{s.d.}(\theta) \ge \frac{2}{a}.$$

Some Simple Results (cont'd)

Result 2

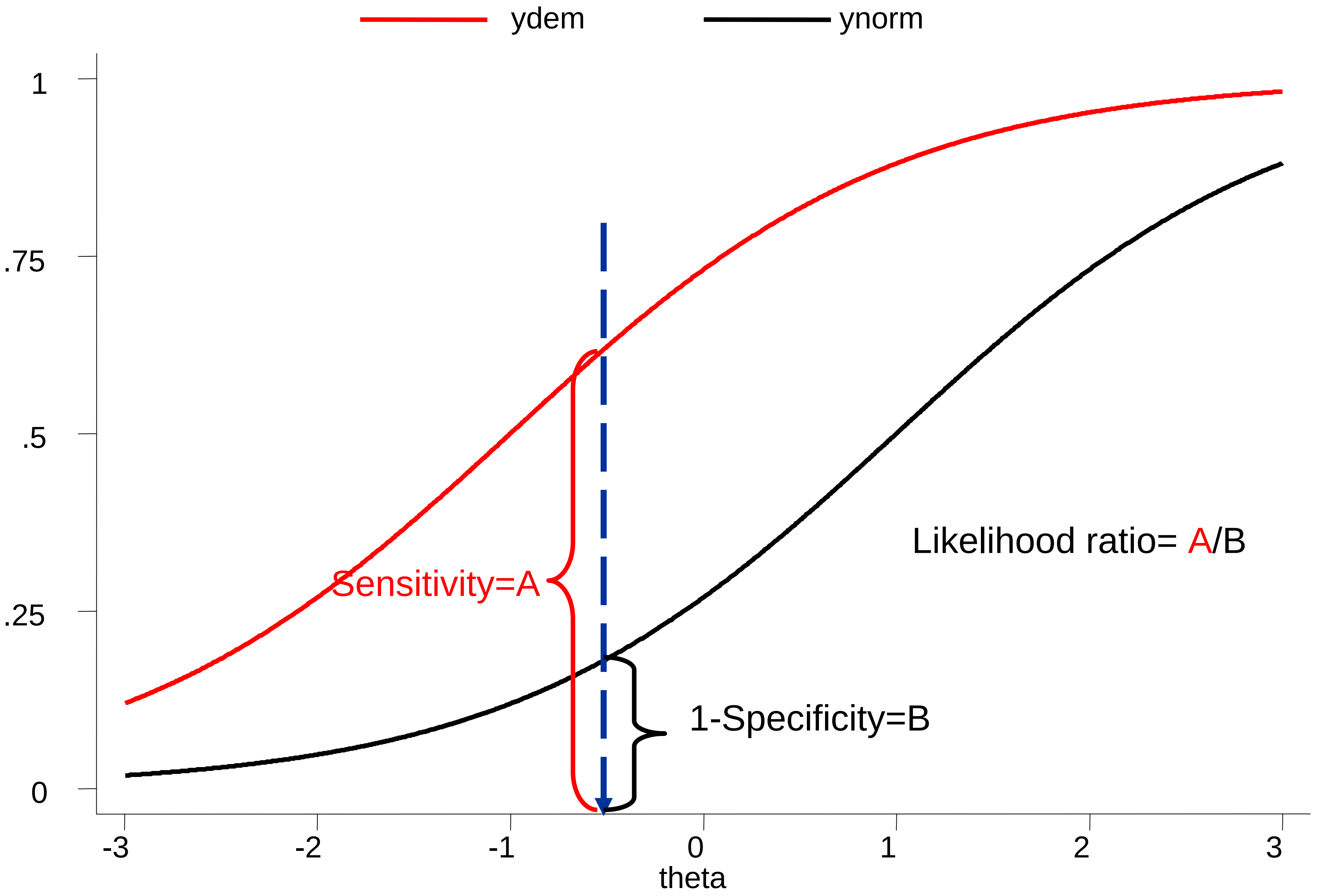
$$\text{Total test information} = I(\theta) = \sum_{j=1}^{k} I_j(\theta) \text{ for k items.}$$

$$I(\theta) = \sum_{j=1}^{k} \frac{1}{[\text{s.d.}_j(\theta)]^2} \leq \frac{1}{4} \sum_{j=1}^{k} a_j^2 \text{ (equality at } P = 0.5)$$

Result 3

$$\text{var}(\theta) = \frac{1}{I(\theta)} = \frac{1}{\sum_{j=1}^{k} \frac{1}{[\text{s.d.}_j(\theta)]^2}} = \frac{1}{k} [\text{Harmonic mean of s.d.}(\theta)_j{}^2]$$

3. IRT and link to sensitivity and specificity



4. Linear model and IRT model—cautions

Logistic Formulation:

$$P(Y=1\,|\,\theta) = \frac{1}{1+\exp[-\{\beta_0+\beta_1\theta\}]}$$

2PL Formulation:

$$P(Y1\,|\,\theta) = \frac{1}{1+\exp[-\{a(\theta-b)\}]}$$

$$= \frac{1}{1+\exp[-\{-ab+a\theta\}]}$$

β_o=-ab

β_1=a

4. Linear model and IRT model—cautions

Becomes even trickier with, say, Uniform DIF:

Logistic regression with uniform DIF:

$$P(Y = 1 | \theta) = \frac{1}{1 + \exp[-\{\beta_0 + G \times \delta + \beta_1 \theta\}]}$$

2PL with uniform DIF:

$$P(Y1 | \theta) = \frac{1}{1 + \exp[-a(\theta - b + G \times \delta)]}$$

$$= \frac{1}{1 + \exp[-\{-ab + a \times G \times \delta + a\theta\}]}$$

5. Measuring change—a simple model

Importance of cognitive change:

1. Clinical interest
2. Research interest
3. Clinical trials of new agents
4. Normal aging

Questions:

1. How to model change in IRT environment?
2. What items are important for detecting change?

The model

$$P(X_{ij}^{B}=1 \text{ and } X_{ij}^{F}=1 \mid \theta_i, \Delta, \alpha_j, \beta_j) = \frac{e^{D\alpha_j(\theta_i-\beta_j)}}{1+e^{D\alpha_j(\theta_i-\beta_j)}} \frac{e^{D\alpha_j(\theta_i+\Delta-\beta_j)}}{1+e^{D\alpha_j(\theta_i+\Delta-\beta_j)}}$$

Assumes two measurements at baseline and final.
2PL formulation with D=1.7 included in model.
Same shift of Δ for every one; as in clinical trial model.
Assume conditional independence given θ_j and Δ.

The result

$$e^{D\alpha_j \Delta} = \frac{P(X_{ij}^{B} = 0 \text{ and } X_{ij}^{F} = 1)}{P(X_{ij}^{B} = 1 \text{ and } X_{ij}^{F} = 0)}$$

$$= \text{Odds ratio for off - diagonal elements}$$

$$\frac{n_{j01}}{n_{j10}} = \frac{\hat{P}(X_{ij}^{B} = 0 \text{ and } X_{ij}^{F} = 1)}{\hat{P}(X_{ij}^{B} = 1 \text{ and } X_{ij}^{F} = 0)}$$

$$= \text{estimate of } e^{D\alpha_j \Delta} \text{ by item j among discordant subjects.}$$

Estimation of Δ

$$\hat{\Delta}_j = \frac{1}{D\hat{\alpha}_j} \ln \frac{n_{j01}}{n_{j10}}$$

can be estimated from each item.

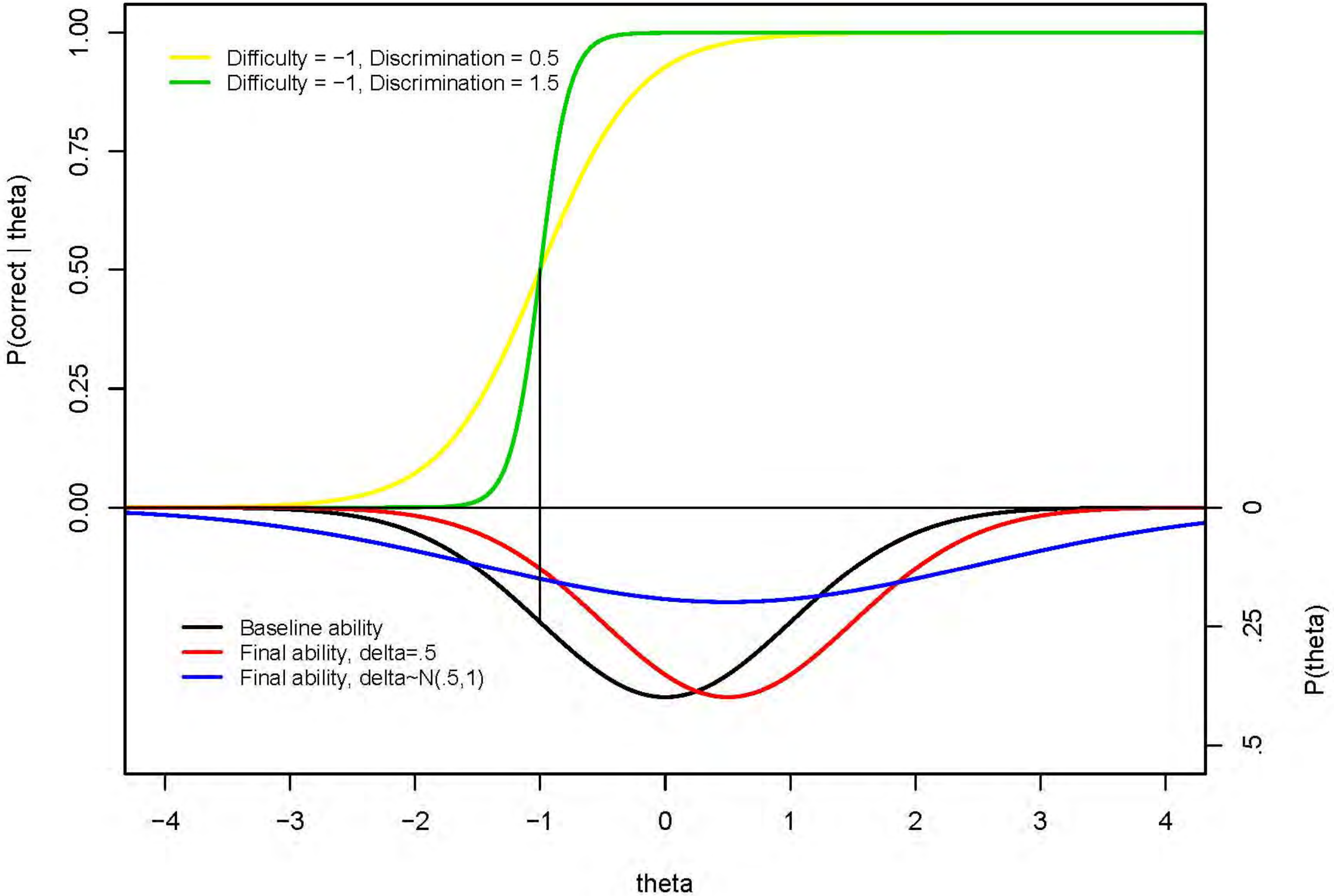
Note that precision depends only on the discrimination. Or does it?

$$v_j = \mathrm{var}(\hat{\Delta}_j) = \left(\frac{1}{D\hat{\alpha}_j}\right)^2 \left(\frac{1}{n_{j01}} + \frac{1}{n_{j10}}\right).$$

It turns out that the off-diagonal frequencies are determined by the difficulty of the item.

Estimation of Δ–continued

1. Estimates can be combined; weighted average
2. We confirmed formulae by simulations and numerical Integration (Run by Doug Tommet)
3. Surprisingly little effect of discrimination and difficulty
4. A picture shows why this is the case
5. The picture also shows the three most important aspects for assessing change



Difficulty = −1, Discrimination = 0.5
Difficulty = −1, Discrimination = 1.5
Baseline ability
Final ability, delta=.5
Final ability, delta~N(.5,1)
P(correct | theta)
1.00
0.75
0.50
0.25
0.00
0
.25
.5
P(theta)
−4
−3
−2
−1
0
1
2
3
4
theta

Change in ability in ACT cohort

ACT = Adult Changes in Thought

Inception cohort of normal elders started 1994

N=2579 at start

Followed every two years

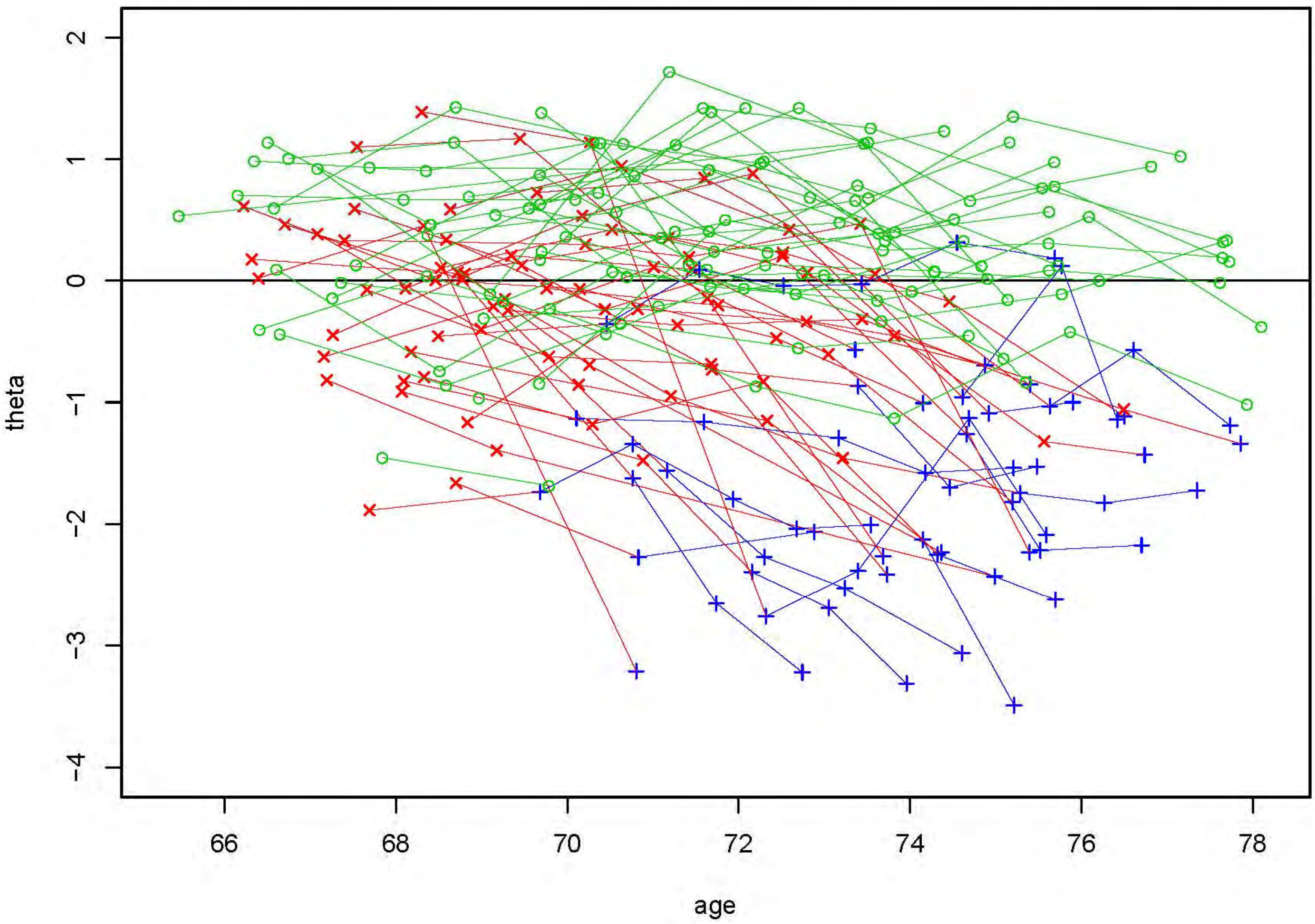
Demented subjects followed every year

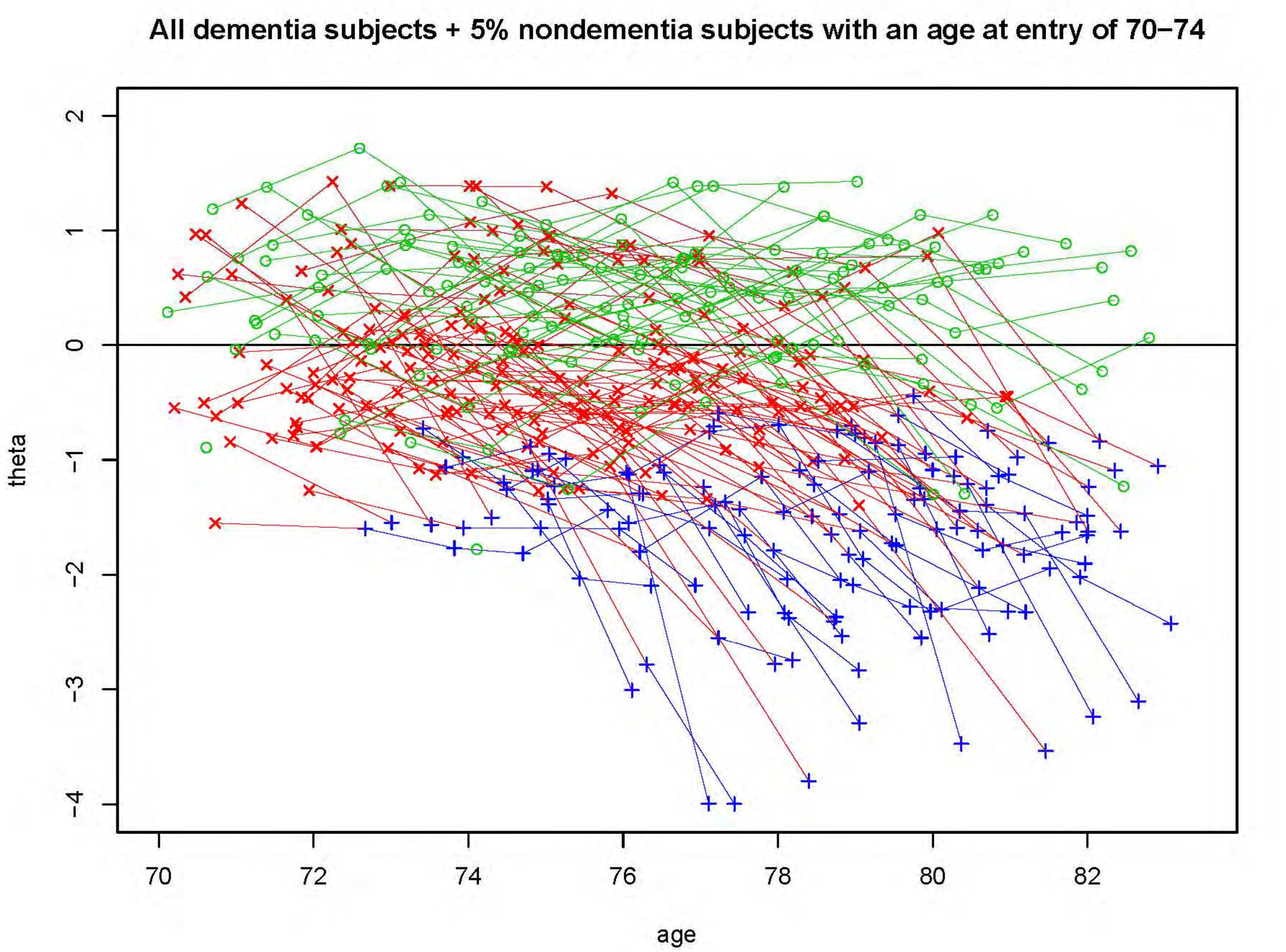
CASI primary instrument for assessing cognition

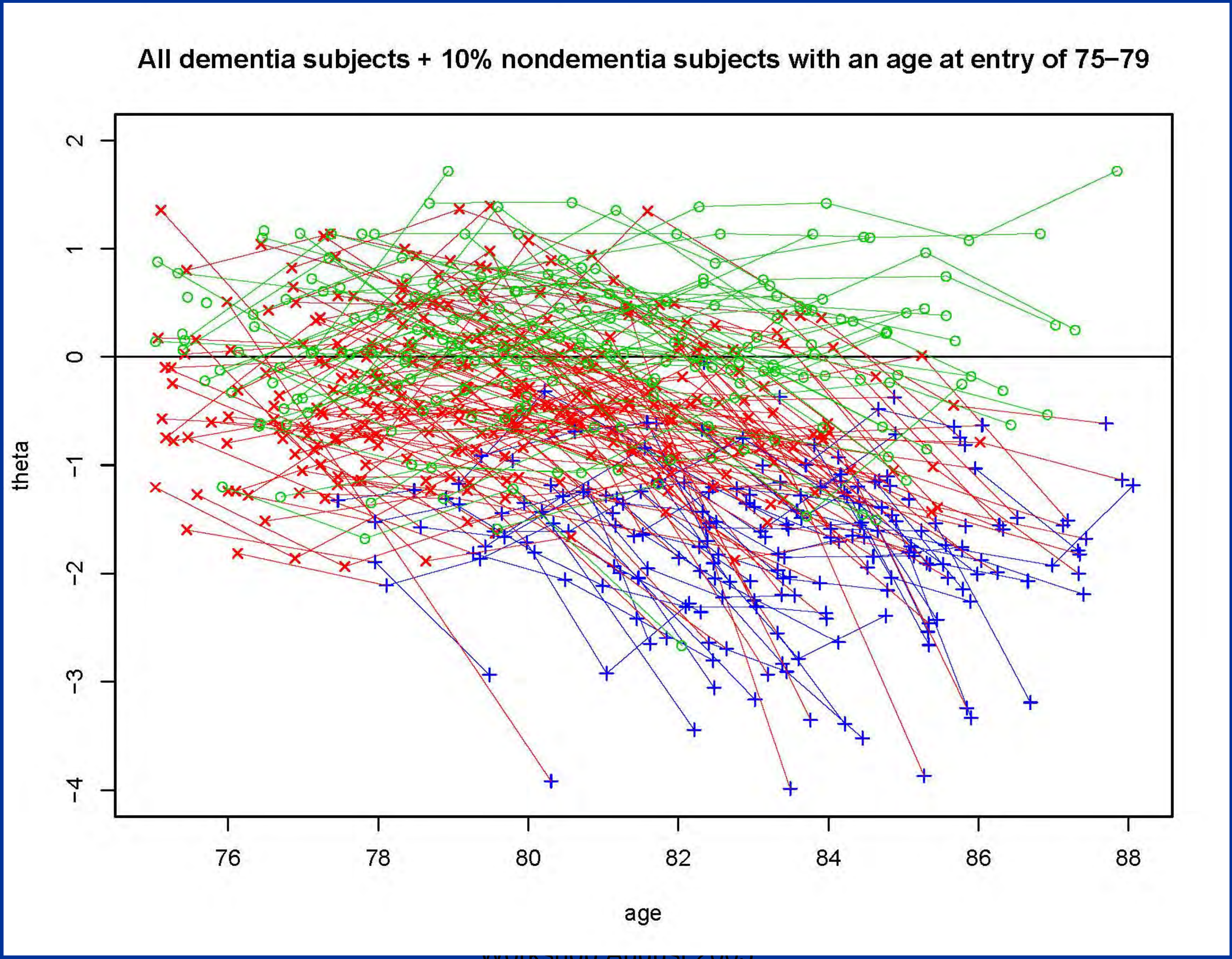
Analytic strategy

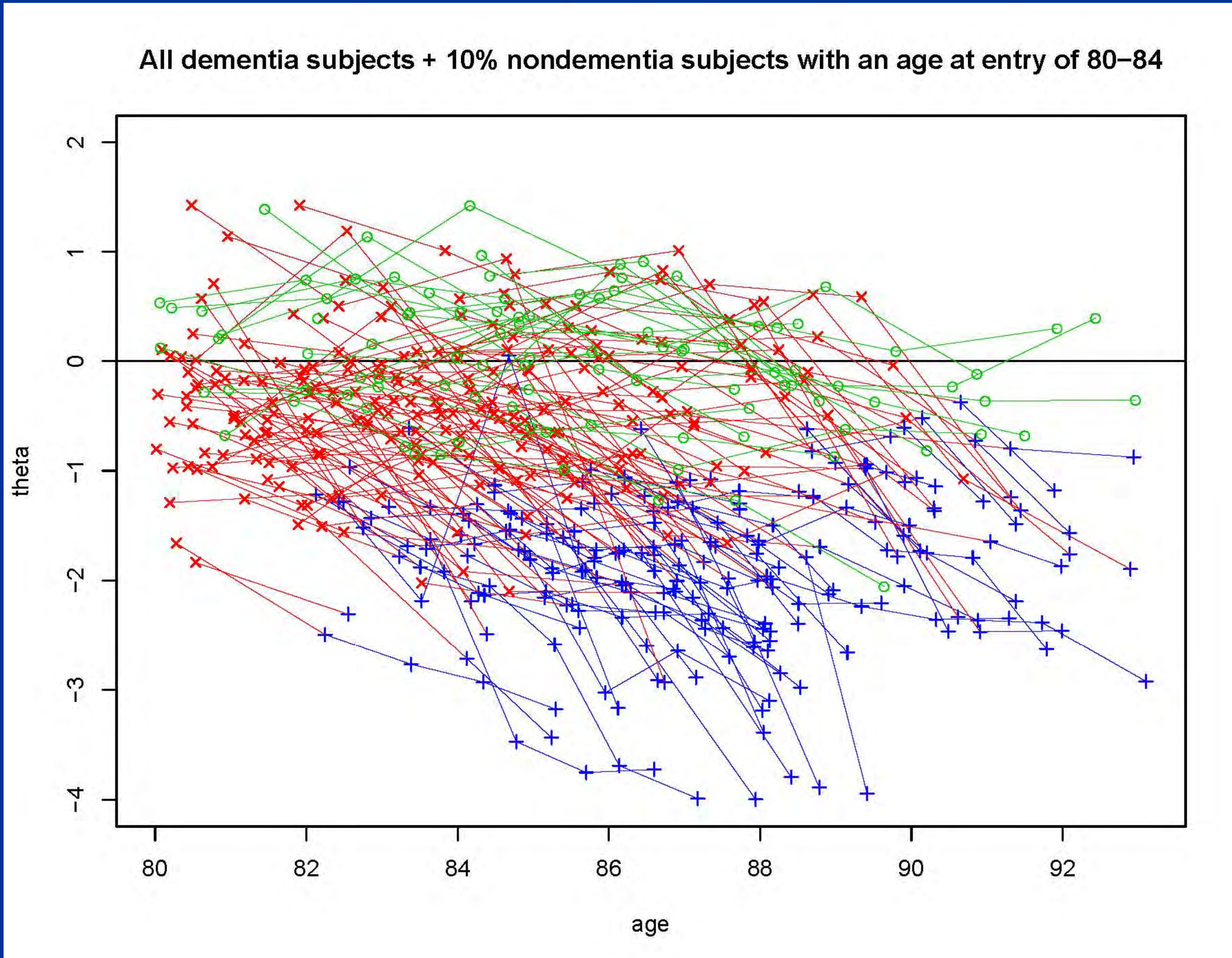
1. Assess cognitive status using PARSCALE
2. Arrange all subjects into one matrix for all times to estimate θ
3. After obtaining θ's we used hierarchical linear model in STATA to analyze change over time
4. In this presentation we look at change over time for particular subgroups
5. Primary purpose is to show Doug Tommet's computing prowess

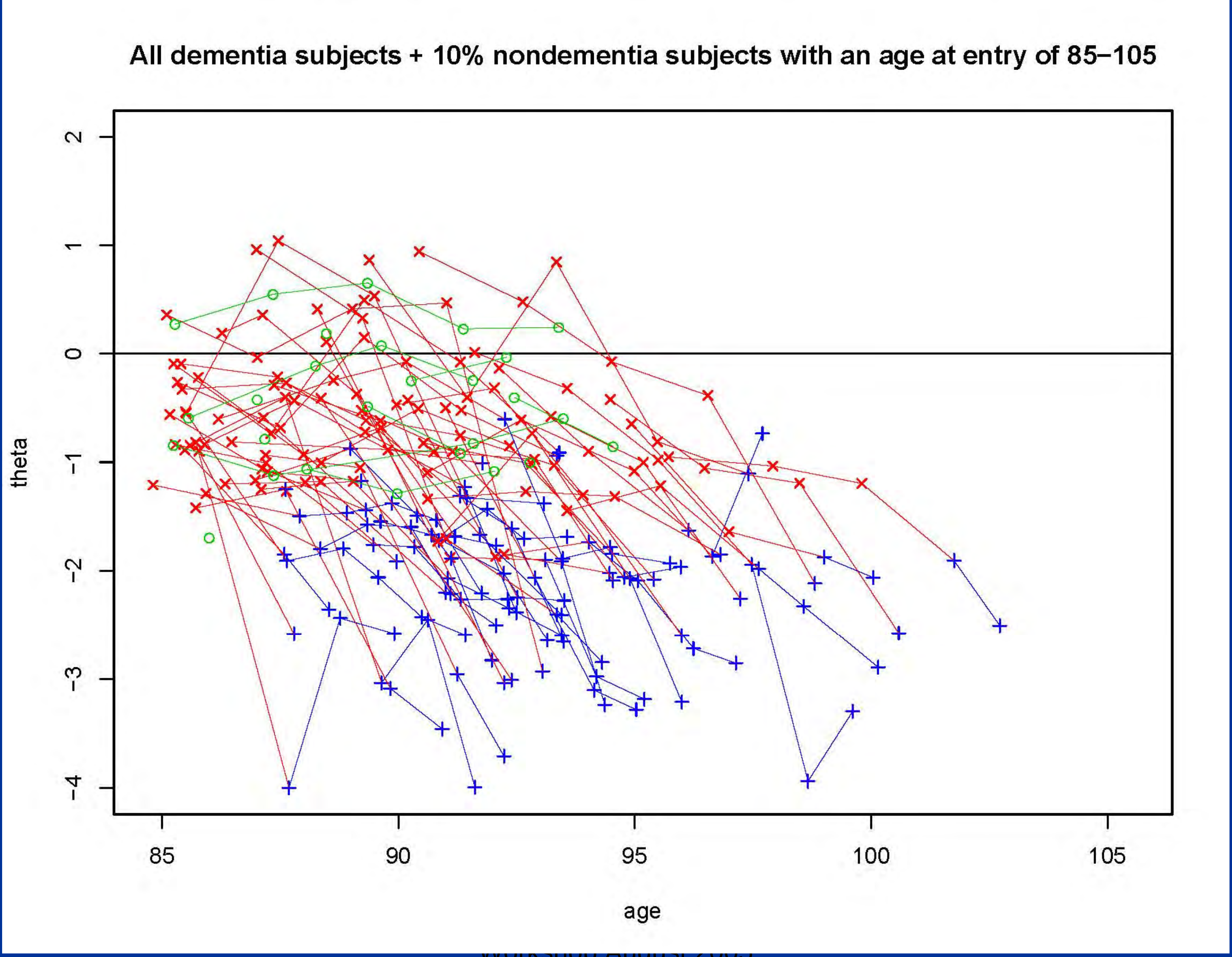
All dementia subjects + 5% nondementia subjects with an age at entry of 65−69

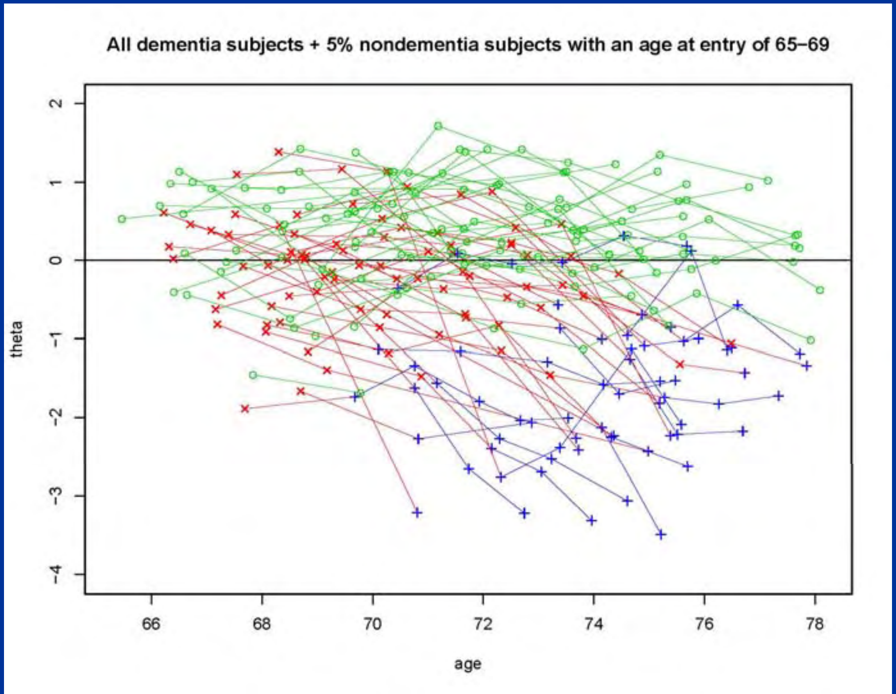


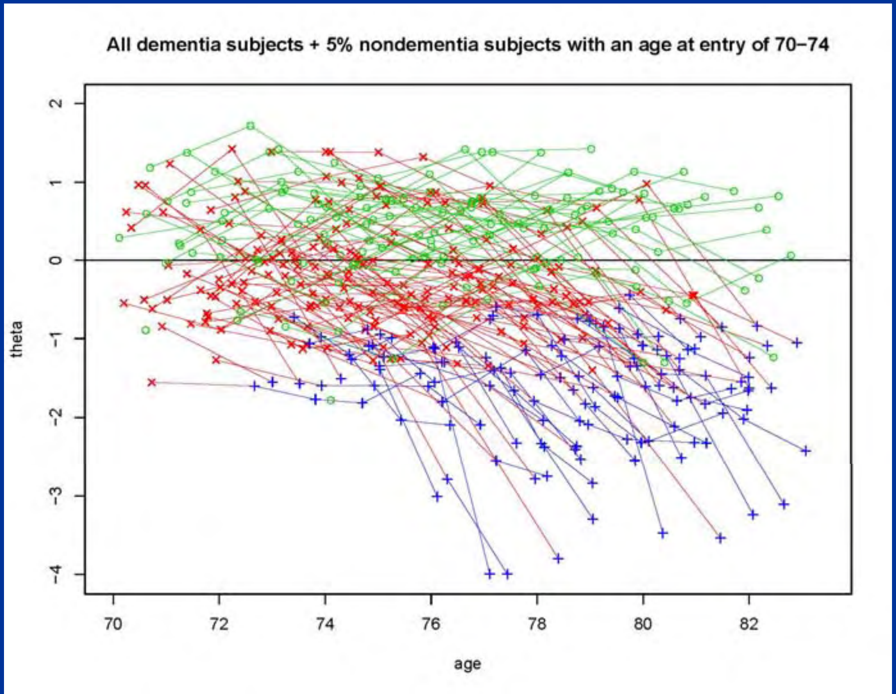

All dementia subjects + 5% nondementia subjects with an age at entry of 70−74
theta
age
2
1
0
−1
−2
−3
−4
70
72
74
76
78
80
82

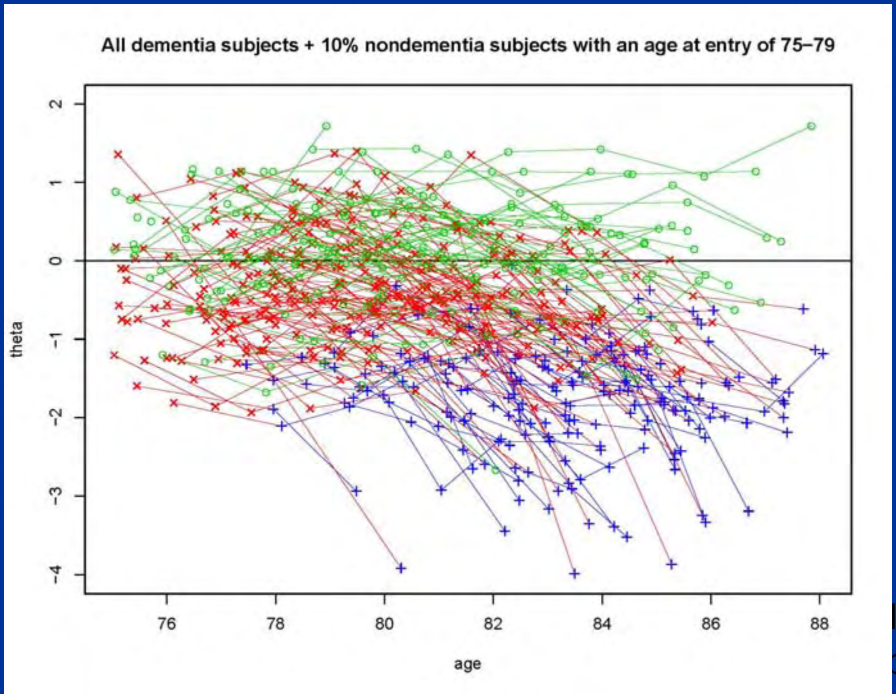

All dementia subjects + 10% nondementia subjects with an age at entry of 75−79
theta
age
2
1
0
−1
−2
−3
−4
76
78
80
82
84
86
88


All dementia subjects + 10% nondementia subjects with an age at entry of 80−84
theta
2
1
0
−1
−2
−3
−4
age
80
82
84
86
88
90
92


All dementia subjects + 10% nondementia subjects with an age at entry of 85−105
theta
age
2
1
0
−1
−2
−3
−4
85
90
95
100
105


All dementia subjects + 5% nondementia subjects with an age at entry of 65–69
theta
age
2
1
0
-1
-2
-3
-4
66
68
70
72
74
76
78


All dementia subjects + 5% nondementia subjects with an age at entry of 70–74
theta
age
2
1
0
-1
-2
-3
-4
70
72
74
76
78
80
82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theta
age
2
1
0
-1
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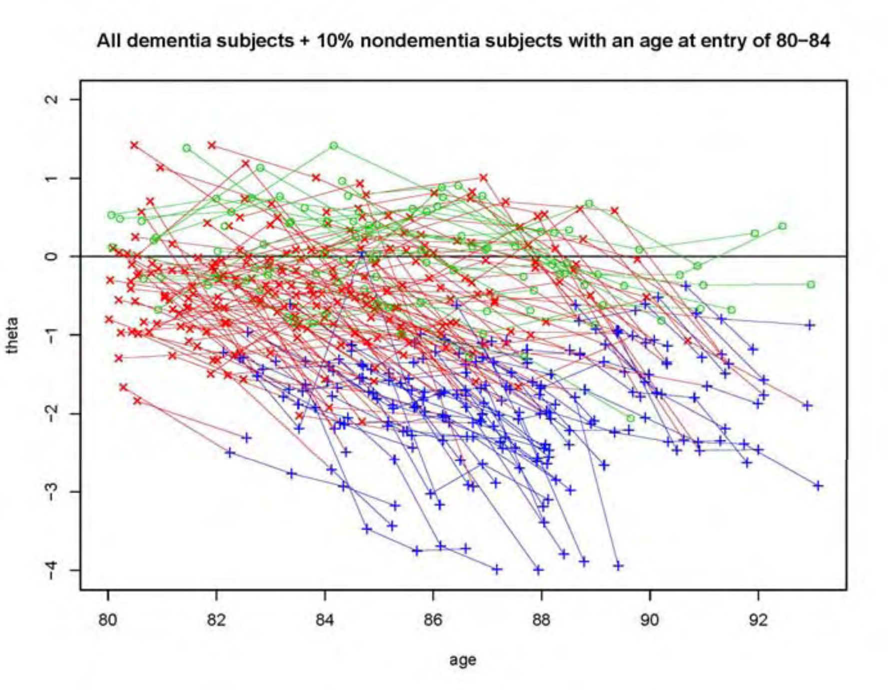
86
88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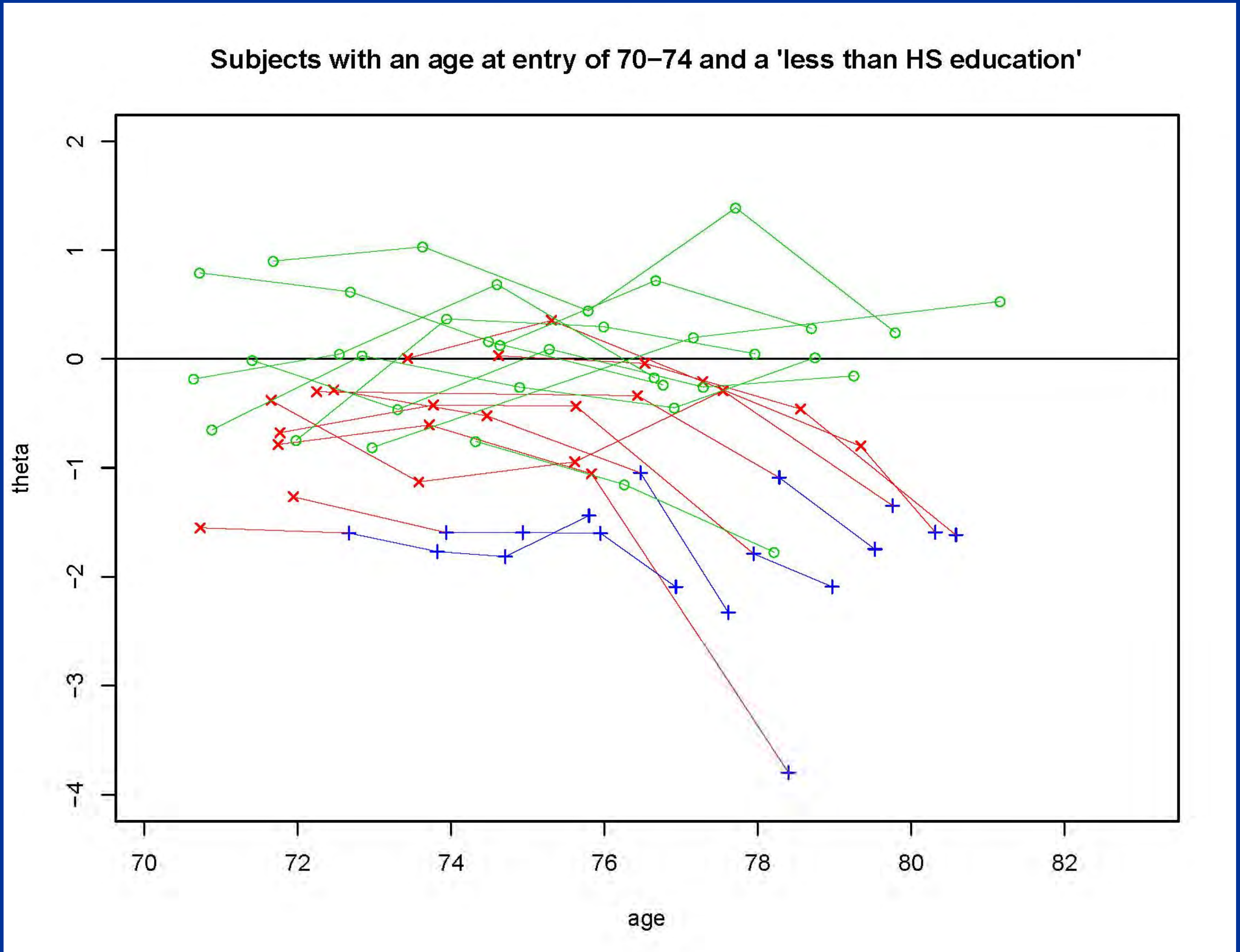
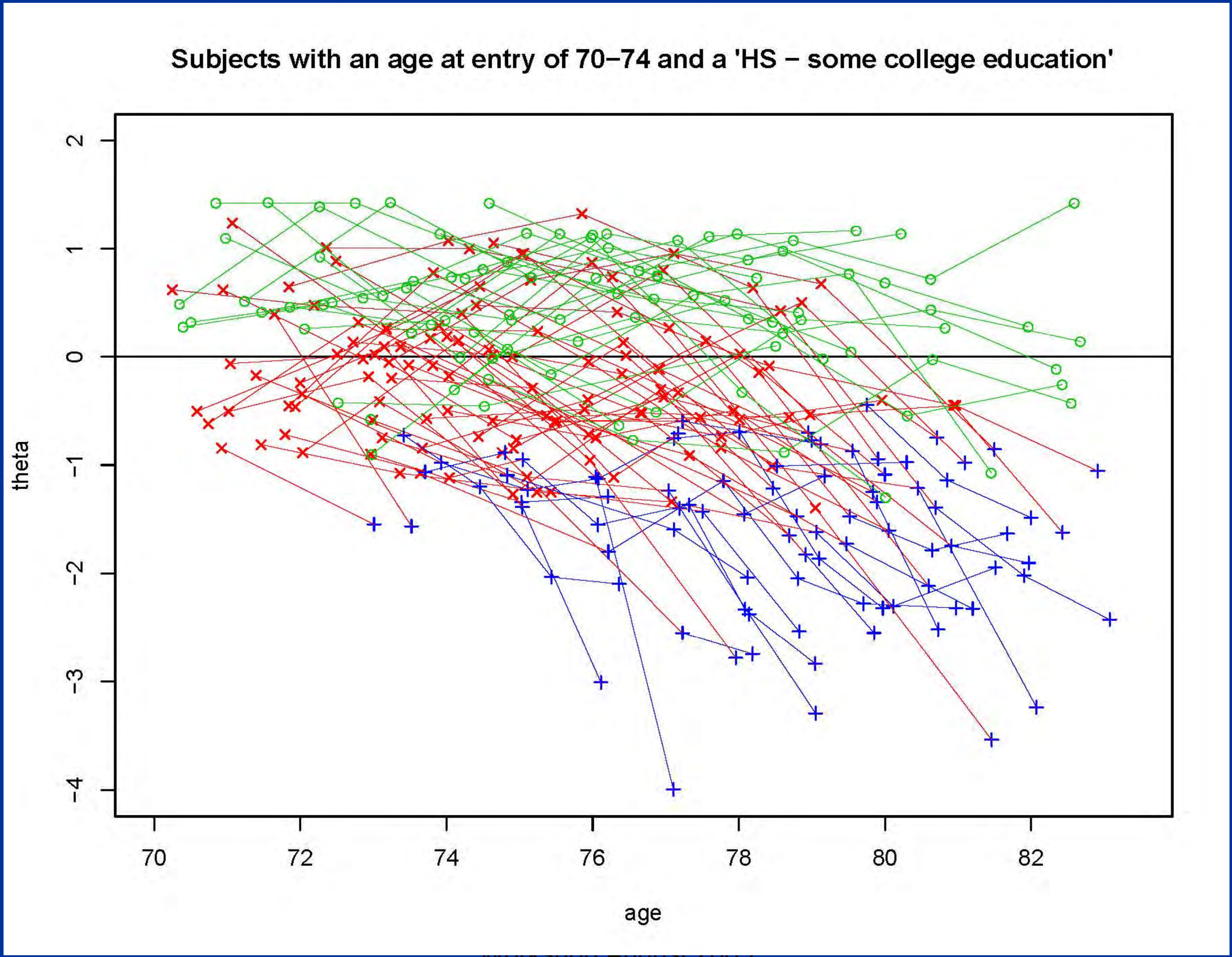
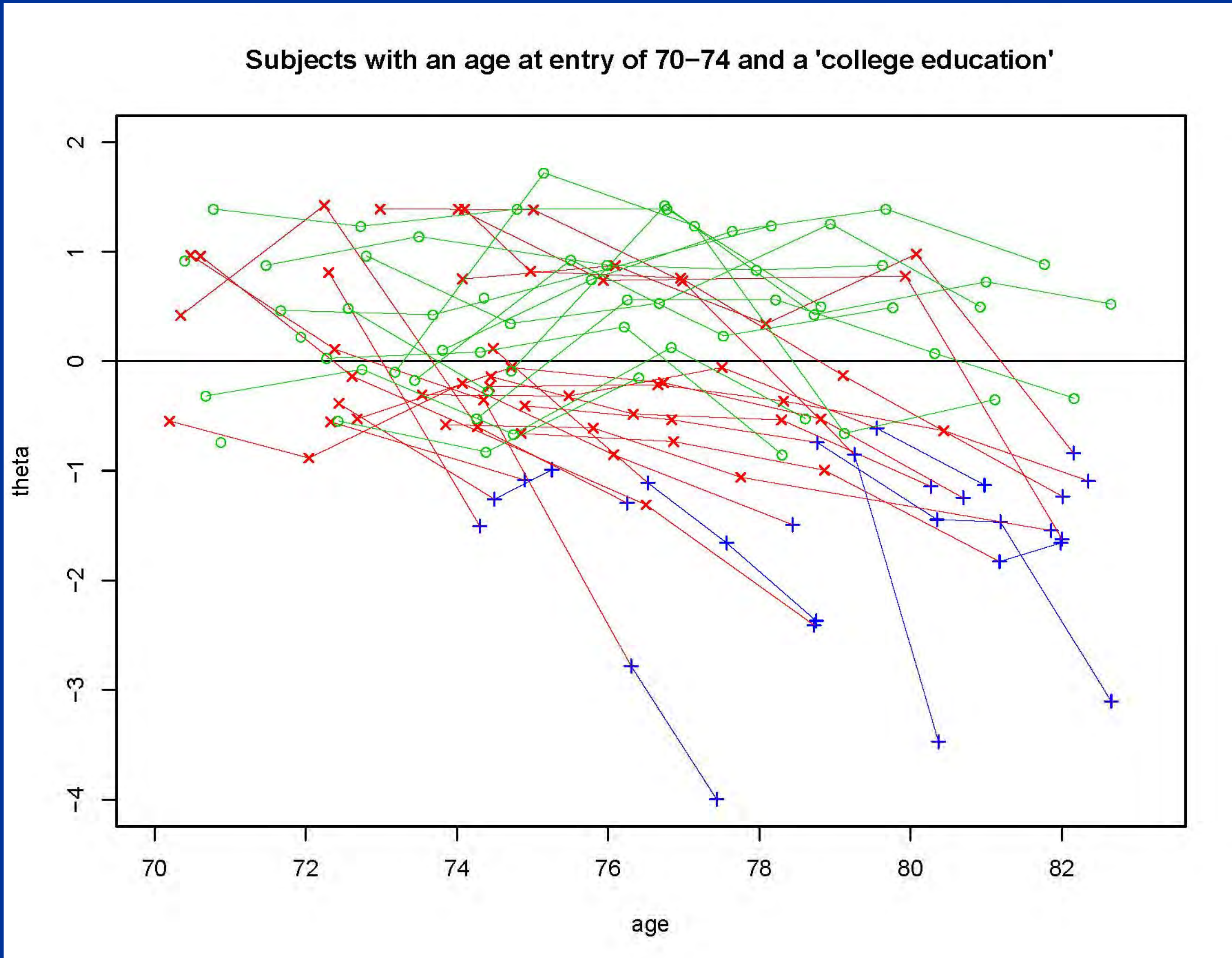
90
92



Subjects with an age at entry of 70−74 and a 'less than HS education'
theta
2
1
0
−1
−2
−3
−4
age
70
72
74
76
78
80
82



Subjects with an age at entry of 70−74 and a 'HS − some college education'
theta
age
2
1
0
−1
−2
−3
−4
70
72
74
76
78
80
82


Subjects with an age at entry of 70−74 and a 'college education'
theta
2
1
0
−1
−2
−3
−4
age
70
72
74
76
78
80
82